Science Supplementary Material **SMT359 RG**
SMT359 Electromagnetism

The Open University

SMT359

Revision Guide

Book 1

Book 1 Chapter 1

Section 1.1 Electric charge is the property that allows particles to exert and experience electromagnetic forces. Electric charge is a scalar quantity which is additive, quantized, locally conserved and invariant.

Section 1.2 Electromagnetic forces are velocity-dependent. This is important for particles moving at speeds comparable to that of light and is also significant in neutral systems containing a large number of slowly-moving particles (e.g. current-carrying wires).

Section 1.3 It is customary to split electromagnetic forces into electric and magnetic contributions. A stationary point charge experiences only the electric force. A moving charge experiences the same electric force as a stationary charge; any additional electromagnetic force that it experiences by virtue of being in motion, rather than being at rest, is classified as a magnetic force.

Section 1.4 Electric forces between stationary charges are called electrostatic forces. The electrostatic force between two charges is given by Coulomb's law:

$$\mathbf{F}_{12} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\,\widehat{\mathbf{r}}_{12},$$

where $\mathbf{F}_{12}$ is the force on particle 1 due to particle 2 and $\widehat{\mathbf{r}}_{12}$ is a unit vector pointing towards particle 1 from particle 2. The electrostatic force due to a number of stationary sources is found by vector addition, using Coulomb's law and the law of addition of force.

Coulomb's law has been experimentally tested over a wide range of length scales. It works well enough for slowly-moving particles and for charges in gaseous media but it is not valid for very rapidly moving particles and its implications can be obscured by the effects of screening or polarization in liquid or solid media.

Section 1.5 The electric field $\mathbf{E}(\mathbf{r})$ is a vector field defined throughout a region of space. Its spatial variation can be visualized using an arrow map or a field line pattern. At any given point the value of the electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{\mathbf{F}}{q},$$

where $\mathbf{F}$ is the force that would be experienced by a charge q placed at the point $\mathbf{r}$. Electric fields obey the principle of superposition: the electric field due to a set of sources is the vector sum of the individual electric fields due to each source. An electric field inherits the symmetry of its sources: any operation that leaves the sources unchanged also leaves the electric field unchanged.

Book 1 Chapter 2

Section 2.1 Charge density is the charge per unit volume. The total charge in a region is the volume integral of the charge density over the region. Electric flux is the surface integral of the electric field over a given surface.

Section 2.2 The integral version of Gauss's law states that the electric flux over a closed surface S is equal to the total charge enclosed by the surface, divided by ε_0. That is,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \frac{Q}{\varepsilon_0} = \frac{1}{\varepsilon_0} \int_V \rho(\mathbf{r})\, \mathrm{d}V,$$

where Q is the charge enclosed by S and V is the volume enclosed by S. This law applies to all distributions of charge (whether stationary or moving) and to all closed surfaces (no matter what their shape). It is unaffected by the presence of charges outside the closed surface. For the special case of stationary charges, Gauss's law can be derived from Coulomb's law, the principle of superposition and the additivity of charge.

Section 2.3 To apply Gauss's law, we exploit the symmetry of the charges to constrain the possible form of the electric field, and choose a suitable closed surface (a Gaussian surface). Ideally, the field has a constant normal component over this surface, or over easily identified faces of the surface. In cases of spherical, cylindrical or planar symmetry it is possible to deduce the electric field from Gauss's law.

Section 2.4 The divergence of the electric field is the electric flux per unit volume. This is a scalar field, represented in Cartesian coordinates by

$$\operatorname{div} \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

The divergence theorem tells us that the surface integral of a vector field over a closed surface S is the volume integral of the divergence of the field over the region V inside S. That is,

$$\int_S \mathbf{E} \cdot \mathrm{d}\mathbf{S} = \int_V \operatorname{div} \mathbf{E}\, \mathrm{d}V.$$

Using this theorem, together with the integral version of Gauss's law, we obtain the differential version of Gauss's law:

$$\operatorname{div} \mathbf{E} = \frac{\rho(\mathbf{r})}{\varepsilon_0}.$$

This applies to all distributions of charge (whether stationary or moving). In highly symmetrical situations it leads to a differential equation which can be solved for the electric field.

Book 1 Chapter 3

Section 3.1 The current I through a surface is a scalar quantity describing the rate at which charge Q crosses the surface: $I = \mathrm{d}Q/\mathrm{d}t$. This is the surface integral of the current density over the surface. The current density $\mathbf{J}(\mathbf{r})$ is a vector field pointing in the direction of current flow. Its magnitude is the magnitude of the current flowing through a tiny plane element perpendicular to the current flow, divided by the area of the element. In microscopic terms, current density is given by $\mathbf{J} = nq\mathbf{v}$, where n is the number density of charge carriers, q is their charge and $\mathbf{v}$ is their drift velocity. To avoid an unlimited accumulation of charge, a current density that is independent of time must be divergence-free: $\operatorname{div} \mathbf{J} = 0$.

Section 3.2 The current element associated with a volume element δV is $\mathbf{J}\,\delta V$, where $\mathbf{J}$ is the current density at the position of the volume element. For a current in a wire, the current element is $I\,\delta\mathbf{l}$, where $\delta\mathbf{l}$ is a directed line element pointing along the wire in the reference direction for current flow.

The magnetic force between two steady current elements is given by the Biot–Savart force law:

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} \frac{I_1\,\delta\mathbf{l}_1 \times (I_2\,\delta\mathbf{l}_2 \times \widehat{\mathbf{r}}_{12})}{r_{12}^2}.$$

This law applies only to steady currents and its use in this book assumes that the magnetic response of materials can be neglected.

Section 3.3 The Biot–Savart law can be split into separate equations describing the production of a magnetic field by a current element and the response of a current element to a magnetic field:

$$\delta\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{I\,\delta\mathbf{l} \times (\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3},$$

$$\delta\mathbf{F} = I\,\delta\mathbf{l} \times \mathbf{B}.$$

Section 3.4 The Lorentz force law states that the total electromagnetic force acting on a point charge q is

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right),$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields at the position of the charge. This law underpins many phenomena and devices, including the Hall effect, Penning traps and cyclotrons.

Book 1 Chapter 4

Section 4.1 The no-monopole law states that the magnetic flux over any closed surface is equal to zero. Using the divergence theorem, it follows that the magnetic field is divergence-free:

$$\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S} = 0 \quad \text{and} \quad \operatorname{div} \mathbf{B} = 0.$$

These equations imply that magnetic field lines have neither starting points nor ending points, but form closed loops. No magnetic monopole has been reliably detected.

Section 4.2 The magnetic circulation around a closed path is the line integral of the magnetic field around the path. The integral version of Ampère's law states that the magnetic circulation around a closed path C is equal to the total current flowing through any open surface S that is bounded by C, multiplied by μ_0, the permeability of free space:

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \mu_0 I = \mu_0 \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}.$$

The sense of positive progression around C and the orientation of S are linked by the right-hand grip rule. Ampère's law applies to all steady current distributions. It does not apply to time-varying currents.

Section 4.3 To apply Ampère's law we use the symmetry of the sources to constrain the form of the magnetic field and choose a suitable closed path. Ideally, the field points in the direction of the closed path and has a constant magnitude around the path, or around individual sections of the path.

Any operation that leaves the source of an electromagnetic field unchanged also leaves the field unchanged. At any point in a plane of reflection, the reflected electric field is obtained by reversing the component of the field *perpendicular* to the plane, leaving the components parallel to the plane unchanged. The reflected magnetic field is obtained by reversing the components of the field *parallel* to the plane, leaving the component perpendicular to the plane unchanged.

Section 4.4 The curl of the magnetic field is a vector field whose components are the magnetic circulation per unit area. In Cartesian coordinates,

$$\operatorname{curl} \mathbf{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\mathbf{e}_z.$$

The curl theorem tells us that the line integral of a vector field around a closed path C is the surface integral of the curl of the field over any surface S that is bounded by C. So,

$$\oint_C \mathbf{B} \cdot \mathrm{d}\mathbf{l} = \int_S \operatorname{curl} \mathbf{B} \cdot \mathrm{d}\mathbf{S}.$$

Using this theorem, together with the integral version of Ampère's law, we obtain the differential version of Ampère's law:

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}.$$

This law applies only to steady current distributions.

Book 1 Chapter 5

Section 5.1 An electrostatic field has zero circulation around any closed loop and is therefore conservative. Conservative fields are always irrotational (i.e. have zero curl) so checking whether the curl of a vector field is equal to zero provides one way of determining whether the field could be an electrostatic field. The conservative nature of electrostatic fields (together with Gauss's law) ensures that there are no electrostatic fields inside an empty conducting cavity.

Section 5.2 The electrostatic potential at a point $\mathbf{r}$ is defined by

$$V(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{E} \cdot \mathrm{d}\mathbf{l} + V_0,$$

where $\mathbf{E}$ is the electrostatic field, $\mathbf{r}_0$ is an arbitrarily chosen reference point and V_0 is the value of the electrostatic potential at the reference point. $V(\mathbf{r})$ is the electrostatic potential energy per unit charge. The change in electrostatic potential energy when a test charge q is displaced from $\mathbf{r}_1$ to $\mathbf{r}_2$ is $q(V(\mathbf{r}_2) - V(\mathbf{r}_1))$.

Any line integral of an electrostatic field can be represented as minus the difference in values of the potential between the end-point and start-point of the path. The electrostatic field corresponding to a given potential is minus the gradient of the potential.

Section 5.3 Electrostatic potentials obey the principle of superposition. In a system of many charges, the total electrostatic potential is the algebraic sum of the electrostatic potentials due to the individual charges. This can be used to find the total electrostatic potential of a collection of charges; the total electrostatic field can then be found by taking minus the gradient of the potential. This method only works for finite distributions of charge.

An electric dipole is a pair of oppositely-charged particles. The dipole potential is an approximation to the electrostatic potential of this arrangement when the distance from the dipole is much greater than the separation of the charges. The dipole potential and the corresponding electrostatic field can be found using the principle of superposition.

Section 5.4 In equilibrium, the electrostatic potential is uniform throughout any conductor. The capacitance of an isolated conductor is the ratio Q/V, where Q is the charge on the conductor and V is the potential of the conductor relative to a zero of potential at infinity. The capacitance of a capacitor is the ratio Q/V, where Q is the charge on the positive plate of the capacitor and V is the potential difference between the positive and negative plates.

The total electrostatic energy stored by a capacitor is

$$U = \frac{1}{2}CV^2.$$

This can be interpreted in terms of the energy stored in the electrostatic field. The energy density of an electric field is $\frac{1}{2}\varepsilon_0 E^2$, where E is the electric field strength. Integrating the energy density over all space gives the total electrostatic energy.

Book 1 Chapter 6

Section 6.1 Steady currents in stationary circuits are driven by non-conservative electric fields. The voltage drop along a stationary path C is the line integral of the electric field along C. The emf in a circuit is the work done per unit charge transferred around the circuit. Most current flows obey Ohm's law, $V = IR$, where V is the voltage drop across a conductor of resistance R carrying a current I.

Section 6.2 The integral version of Faraday's law states that

$$\oint_C \mathbf{E} \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}}{\mathrm{d}t}\int_S \mathbf{B} \cdot \mathrm{d}\mathbf{S},$$

that is, the induced emf around a stationary closed loop C is the rate of decrease of the magnetic flux over an open surface S bounded by C. The positive sense of progression around C and the orientation of S are linked by the right-hand grip rule. The minus sign in Faraday's law is consistent with Lenz's law, which states that an induced current flows in such a way that its magnetic flux opposes the *change* in the magnetic flux that produced it.

Section 6.3 The differential version of Faraday's law states that

$$\operatorname{curl}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

where $\mathbf{E}$ is the electric field at a given place and time, and $\mathbf{B}$ is the magnetic field at the same place and time. If a magnetic field is independent of time in a simply-connected region, the electric field is conservative there.

Section 6.4 The voltage drop along a moving path C is the line integral of $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ along the path. This voltage drop appears in Ohm's law for the moving circuit: $V_{\text{drop}} = IR$. The induced emf around a moving circuit is the line integral of $\mathbf{E} + \mathbf{v} \times \mathbf{B}$ around the circuit. The generalized Faraday law states that the emf around a moving circuit is equal to the rate of decrease of magnetic flux through the circuit.

Book 1 Chapter 7

Section 7.1 The law of conservation of charge applies locally at each point and time, so any variation of the total charge within a closed surface must be due to charges that flow across the surface of the region. This principle leads to the equation of continuity:

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \mathbf{J} = 0,$$

where ρ is the charge density and $\mathbf{J}$ is the current density at any given point and time. In magnetostatic situations, $\partial\rho/\partial t = \operatorname{div} \mathbf{J} = 0$.

Section 7.2 Ampère's law, $\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J}$, is a law of magnetostatics. It applies when $\partial\rho/\partial t = \operatorname{div} \mathbf{J} = 0$. The appropriate generalization, valid for time-dependent charge and current densities, is the Ampère–Maxwell law:

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}.$$

The extra term, $\varepsilon_0 \mu_0 \partial \mathbf{E}/\partial t$, on the right-hand side is called the Maxwell term.

Section 7.3 Maxwell's four equations

$$\operatorname{div} \mathbf{E} = \frac{\rho}{\varepsilon_0}$$

$$\operatorname{div} \mathbf{B} = 0$$

$$\operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{curl} \mathbf{B} = \mu_0 \mathbf{J} + \varepsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

describe the dynamical behaviour of electromagnetic fields. They are the same in all inertial frames of reference and are unaffected by time-reversal. They are not valid in rotating frames of reference.

Section 7.4 An electromagnetic wave is an oscillating disturbance of electric and magnetic fields that propagates in accordance with Maxwell's equations. We concentrate on linearly polarized monochromatic plane waves. In empty space, the electric and magnetic waves are in phase with one another, with $B = E/c$. They are mutually perpendicular and transverse to the direction of propagation. In empty space, electromagnetic waves travel at speed $c = 1/\sqrt{\varepsilon_0 \mu_0} = 3.00 \times 10^8\,\mathrm{m\,s^{-1}}$.

Electromagnetic waves with frequencies in the visible range, 4×10^{14} Hz to 8×10^{14} Hz, are called light, but the known electromagnetic spectrum also embraces radio waves, microwaves, infrared, ultraviolet, X-rays and gamma rays. Electromagnetic waves transport energy. The amount of energy carried by the magnetic wave is the same as that carried by the electric wave. The energy flux is the total energy transported per unit area per unit time across a plane area perpendicular to the direction of propagation of the electromagnetic wave. Averaging over a complete cycle,

$$\text{average energy flux} = \frac{1}{2}\sqrt{\frac{\varepsilon_0}{\mu_0}} E_0^2,$$

where E_0 is the amplitude of the electric wave.

Book 2

Book 2 Chapter 2

Section 2.1 In the presence of media, Maxwell's equations need to take account of charges and currents due to atoms. Charges $+q$ and $-q$ separated by a displacement $\mathbf{d}$ form an electric dipole, which has dipole moment $\mathbf{p} = q\mathbf{d}$. Electric fields induce a dipole moment in each atom by displacing the electron cloud relative to the nucleus. Asymmetric molecules generally have permanent dipole moments, and an electric field exerts a torque on such molecules that tends to align them preferentially in the direction of the field. However, alignment with the field is disrupted by thermal motion, and is small if the electric potential energy of the dipole pE is small compared with $k_{\mathrm{B}}T$.

Section 2.2 The polarization $\mathbf{P}$ is the electric dipole moment per unit volume: $\mathbf{P} = n\langle\mathbf{p}\rangle$, where n is the number density of dipoles and $\langle\mathbf{p}\rangle$ is the average dipole moment per molecule. The electric susceptibility χ_E

is defined by $\mathbf{P} = \chi_E \varepsilon_0 \mathbf{E}$. For linear, isotropic, homogeneous (LIH) materials, χ_E is independent of $\mathbf{E}$, polarization $\mathbf{P}$ is in the same direction as $\mathbf{E}$, and χ_E is independent of position. The value of χ_E depends on the material and on the temperature.

Section 2.3 The polarization of the molecules in a dielectric is equivalent to a volume charge density $\rho_\mathrm{b} = -\operatorname{div} \boldsymbol{P}$, together with a surface charge density $\sigma_\mathrm{b} = \mathbf{P} \cdot \widehat{\mathbf{n}}$, where $\widehat{\mathbf{n}}$ is an outward-pointing unit normal to the surface.

Section 2.4 The electric displacement, $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$, is directly related to the free charge density ρ_f by the form of Gauss's law that is used for materials: $\operatorname{div} \boldsymbol{D} = \rho_\mathrm{f}$. The integral version of this relationship is $\int_S \mathbf{D} \cdot \mathrm{d}\mathbf{S} = \int_V \rho_\mathrm{f} \, \mathrm{d}V$. For an LIH material, $\mathbf{D} = \varepsilon \varepsilon_0 \mathbf{E}$, where ε is the relative permittivity of the material.

Section 2.5 At the interface between dielectric materials 1 and 2, the fields $\mathbf{D}$ and $\mathbf{E}$ satisfy the boundary conditions $D_{1\perp} = D_{2\perp}$ and $E_{1\parallel} = E_{2\parallel}$.

Section 2.6 For time-dependent fields, the relative permittivity of materials decreases as the frequency of variation of the field increases, initially because the permanent dipoles cannot reorient fast enough, and then because of the inertia of the electron clouds. Inclusion of polarization effects in the Ampère–Maxwell equation leads to the form of the equation appropriate for dielectric materials: $\operatorname{curl} \boldsymbol{B} = \mu_0 (\mathbf{J}_\mathrm{f} + \partial \mathbf{D}/\partial t)$.

Book 2 Chapter 3

Section 3.1 A current I flowing in a loop with area ΔS has magnetic dipole moment $\mathbf{m} = |I| \, \Delta\mathbf{S}$, where the direction of the oriented area $\Delta\mathbf{S}$ is related to the current circulation by a right-hand grip rule. Magnetic fields tend to align permanent dipole moments that are present, and they induce dipole moments in all atoms in a sense that opposes the applied field.

Section 3.2 The magnetization $\mathbf{M}$ is the net magnetic dipole moment per unit volume, and is related to the number density of dipoles n and the average magnetic moment per dipole $\langle \mathbf{m} \rangle$ by $\mathbf{M} = n \langle \mathbf{m} \rangle$. Diamagnetism occurs in materials whose molecules have no permanent magnetic moments. In an applied field, these molecules acquire induced moments in the opposite direction to the applied field, so $\mathbf{M}$ is in the opposite direction to $\mathbf{B}$. Paramagnetism occurs in materials whose molecules have permanent magnetic moments. Alignment of the magnetic dipoles with the field is frustrated by thermal motion and collisions, and the magnetization increases as the ratio of magnetic potential energy to thermal energy, $mB/k_\mathrm{B}T$, increases. The resulting magnetization is parallel to the applied field. In ferromagnetic materials, permanent magnetic dipoles spontaneously align in the same direction within small domains. Application of an increasing external field causes the magnetization of different domains to become progressively more aligned with the field.

The magnetic susceptibility χ_B is defined by $\mathbf{M} = \chi_B \mathbf{B}/\mu_0$, where χ_B is small and negative for diamagnetic materials, and small and positive for paramagnetic materials. For linear, isotropic, homogeneous (LIH) materials, which include most diamagnetic and paramagnetic substances, χ_B is independent of $\mathbf{B}$, $\mathbf{M}$ is in the same direction as $\mathbf{B}$ if χ_B is positive and in the opposite direction to $\mathbf{B}$ if χ_B is negative, and χ_B is independent of position. For non-linear materials (e.g. ferromagnetic substances), χ_B depends on $\mathbf{B}$ and on the past history of the specimen.

Section 3.3 The magnetization of the atoms or molecules in a material is equivalent to a bound current density $\mathbf{J}_\mathrm{b} = \operatorname{curl} \boldsymbol{M}$, together with a bound surface current per unit length $\mathbf{i}_\mathrm{b} = \mathbf{M} \times \widehat{\mathbf{n}}$, where $\widehat{\mathbf{n}}$ is the outward normal to the surface.

Section 3.4 The magnetic intensity, $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$, is related to the free current density $\mathbf{J}_\mathrm{f}$ by the versions of Ampère's law used for materials:

$$\operatorname{curl} \boldsymbol{H} = \mathbf{J}_\mathrm{f} \quad \text{and} \quad \oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \mathbf{J}_\mathrm{f} \cdot \mathrm{d}\mathbf{S} = I_\mathrm{f}.$$

The relative permeability μ of a material is defined by $\mathbf{B} = \mu \mu_0 \mathbf{H}$. For LIH materials, μ is independent of $\mathbf{H}$ and position. For ferromagnetic materials,
μ depends on H.

The Ampère–Maxwell law can also be expressed in terms of $\mathbf{H}$:

$$\operatorname{curl} \boldsymbol{H} = \mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t} \quad \text{and} \quad \oint_C \mathbf{H} \cdot \mathrm{d}\mathbf{l} = \int_S \left(\mathbf{J}_\mathrm{f} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot \mathrm{d}\mathbf{S}.$$

Section 3.5 At the interface between two materials, 1 and 2, the fields $\mathbf{B}$ and $\mathbf{H}$ satisfy the boundary conditions $B_{1\perp} = B_{2\perp}$ and $H_{1\parallel} = H_{2\parallel}$.

Section 3.6 In an electromagnet, an **H** field is generated by passing a current through a coil. The **H** field is concentrated in the region inside the coil. The **B** field is amplified by (almost) filling the coil with a ferromagnetic core. Since the normal component of **B** is continuous, a large **B** field is produced in the air outside the ends of the ferromagnetic material.

Book 2 Chapter 4

Section 4.1 Coulomb's law can be used to calculate the electric field and electrostatic potential if the distribution of charge is known. It is often more straightforward to calculate the (scalar) electrostatic potential first, and then calculate the (vector) electric field using the relationship $\mathbf{E} = -\operatorname{grad} V$. For problems with high symmetry, the electric field can sometimes be calculated using the integral version of Gauss's law.

Section 4.2 In a region containing a linear isotropic homogeneous (LIH) dielectric, the electrostatic potential V satisfies Poisson's equation,

$$\nabla^2 V = -\frac{1}{\varepsilon\varepsilon_0}\rho_{\mathrm{f}},$$

where ρ_{f} is the free charge density and ε is the relative permittivity. In any region where $\rho_{\mathrm{f}} = 0$, Poisson's equation reduces to Laplace's equation,

$$\nabla^2 V = 0.$$

The operator ∇^2 is known as the Laplacian, and is equivalent to the combined div grad operation. In Cartesian coordinates,

$$\nabla^2 V = \operatorname{div}(\operatorname{grad} V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}.$$

In systems with a high degree of symmetry, it may be possible to solve Poisson's equation, or Laplace's equation, by direct integration. At a boundary between two dielectric materials, the potential must remain the same across the boundary, and the perpendicular component of $\varepsilon \operatorname{grad} V$ must remain constant.

Section 4.3 If the value of V is specified on the boundaries of a region, then the solution of Poisson's equation, or Laplace's equation, that matches the boundary conditions is the unique solution. This means that if a solution is found, by whatever means, we can be confident that it is the only solution.

The method of images can sometimes be used to determine the electrostatic potential and field in a system consisting of point, or line, charges and conducting surfaces. The conducting surfaces are replaced by image charges in such a way that the potential and field match the boundary conditions for the original system. Another method is to use solutions to Laplace's equation that have the appropriate symmetry to try to match the boundary conditions for a particular situation.

Section 4.4 In many cases, Laplace's and Poisson's equations cannot be solved analytically, and numerical methods are used. An important type of numerical method for the solution of such differential equations is the finite difference method. The region of interest is divided up by a grid, and simultaneous equations are set up for the potential V at each grid point in terms of the potential at neighbouring points. These equations are solved iteratively, constrained by the values of V on the boundary grid points.

Book 2 Chapter 5

Section 5.1 The relationship between steady currents and magnetic fields is described by Ampère's law:

$$\oint_C \mathbf{B}\cdot \mathrm{d}\mathbf{l} = \mu_0 \int_S \mathbf{J}\cdot \mathrm{d}\mathbf{S} \quad \text{and} \quad \operatorname{curl}\boldsymbol{B} = \mu_0\mathbf{J}.$$

The integral version of this law is useful for calculating magnetic fields produced by currents in situations with high symmetry.

Section 5.2 The Biot–Savart law relates the magnetic field to a line integral over a current in a circuit (or a volume integral of a current density):

$$\mathbf{B} = \frac{\mu_0 I}{4\pi}\int_{\text{circuit}} \frac{\delta\mathbf{l}' \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}.$$

This expression can be used to derive the field of a line current segment, and this is useful for calculating fields due to current loops of complex shapes, and particularly valuable for computer-based calculations.

The existence of a magnetic vector potential $\mathbf{A}$ is a consequence of the no-monopole equation. $\mathbf{A}$ is defined by the equation $\mathbf{B} = \operatorname{curl} \boldsymbol{A}$.

Analytic solutions for the field generated by current distributions are generally not possible, and computers are programmed to find solutions using numerical methods.

Section 5.3 Cellular currents in biological organisms produce potential differences that are measured on the surface of the body and (bio)magnetic fields that are measured outside the body. Both provide useful clinical information, but biomagnetic measurements have the advantages that the field is not affected by intervening tissue and there need be no contact between sensing device and the body. The Biot–Savart equation can be used as a simple model of the field produced by intra-cellular currents. This has medical applications in imaging human heart and brain functions.

Section 5.4 The calculation of currents from a knowledge of the associated magnetic fields is known as the magnetic inverse problem. In general, unique solutions to the magnetic inverse problem are not possible unless assumptions are made restricting possible current configurations.

Book 2 Chapter 6

Section 6.1 The Lorentz force on a particle with charge q moving at velocity $\mathbf{v}$ in an electric field $\mathbf{E}$ and a magnetic field $\mathbf{B}$ is given by $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$. Only the electric field part of the Lorentz force, $q\mathbf{E}$, can change the kinetic energy of a charged particle; the magnetic part, $q\mathbf{v} \times \mathbf{B}$, is perpendicular to the velocity and therefore cannot affect the speed or the kinetic energy of the particle.

Section 6.2 In an electric field, a charged particle with mass m has an acceleration $q\mathbf{E}/m$. In uniform electric fields, a particle's motion can be determined by using the constant acceleration equations. Electric fields are used to accelerate and steer electrons in many important devices, such as the scanning electron microscope.

In a uniform magnetic field, a charged particle has an acceleration perpendicular to the field and to its velocity, and it therefore travels in a circular orbit in a plane perpendicular to the field, or follows a helical path around a field line. The angular frequency and radius of the orbit are known as the cyclotron frequency, $\omega_\mathrm{c} = |q|B/m$, and the cyclotron radius, $r_\mathrm{c} = mv_\perp/|q|B$. The particle's motion can be described by a velocity component parallel to the magnetic field $v_\parallel$ and a perpendicular speed $v_\perp$, and in a uniform magnetic field both $v_\perp$ and $v_\parallel$ remain constant. Combinations of electric and magnetic fields can be used to focus beams of charged particles.

Section 6.3 Where the direction of a magnetic field changes, a transverse drift is superimposed on the spiralling motion of charged particles. The direction of drift is perpendicular to the field direction and to a vector drawn to the field line from its centre of curvature. When the field strength depends on position, the orbit drifts perpendicular to the field direction and to the direction of the gradient of the field magnitude.

A magnetic bottle is a region of non-uniform field that can be used to trap charged particles. The Earth's magnetic field acts as a magnetic bottle for electrons and protons in the Van Allen belts. Transverse drift of these particles produces the Earth's ring current.

Book 2 Chapter 7

Section 7.1 For many materials, current density $\mathbf{J}$ is proportional to electric field, so $\mathbf{J} = \sigma\mathbf{E}$, where σ is the electrical conductivity. This local relationship leads to the result $V = IR$ for the voltage drop V across a conducting object when current I flows through it. R is the resistance of the object. Both of these relationships are known as Ohm's law. More generally, in the presence of magnetic fields, $\mathbf{J} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

Measurements of the resistance of the ground can be used to deduce information about underlying structure.

For steady currents, the potential V within a conducting material obeys Laplace's equation, $\nabla^2 V = 0$. If this equation is solved, with appropriate boundary conditions, the current density and electric field can be calculated using $\mathbf{E} = -\operatorname{grad} V = \mathbf{J}/\sigma$. The current flowing can then be determined by integrating the current density, $I = \int_S \mathbf{J} \cdot \mathrm{d}\mathbf{S}$, and the resistance calculated using $R = \Delta V/I$.

Section 7.2 Emfs can be induced in a circuit either by moving the circuit with respect to a stationary magnetic field, or by a time-dependent magnetic field, with no movement of the circuit, or by a combination of both. In all cases, the emf is given by

$$V_\mathrm{emf} = \oint_C (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathrm{d}\mathbf{l} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t},$$

where $\mathbf{v}$ is the velocity of an element $\mathrm{d}\mathbf{l}$ of the circuit defined by the path C, and Φ is the flux through a surface S bounded by the path C.

The mutual inductance M between two circuits, 1 and 2, is defined by $M = \mathrm{d}\Phi_{21}/\mathrm{d}I_1$, where Φ_{21} is the flux in circuit 2 due to a current I_1 in circuit 1. The self-inductance L for a circuit in which current I produces flux Φ is defined by $L = \mathrm{d}\Phi/\mathrm{d}I$. In the absence of ferromagnetic media, these relationships simplify to $M = \Phi_{21}/I_1$ and $L = \Phi/I$. The unit of inductance is the henry (symbol H). The emf induced in an inductance is given by $V_{\mathrm{emf}} = -\mathrm{d}\Phi/\mathrm{d}t = -L\,\mathrm{d}I/\mathrm{d}t$.

Book 2 Chapter 8

Section 8.1 The potential energy of a system of point charges can be expressed in terms of the values of the charges and the potentials at the positions where they are located:

$$U = \tfrac{1}{2}\sum_{i=1}^{n} Q_i V_i.$$

For charge distributed over surfaces and throughout a volume, the potential energy is given by

$$U = \tfrac{1}{2}\int_S \sigma(\mathbf{r})\,V(\mathbf{r})\,\mathrm{d}S + \tfrac{1}{2}\int_\tau \rho(\mathbf{r})\,V(\mathbf{r})\,\mathrm{d}\tau.$$

The energy stored in a capacitor is

$$U = \tfrac{1}{2}CV^2 = \frac{Q^2}{2C} = \tfrac{1}{2}QV.$$

Alternatively, the electric potential energy can be associated with the electric field. The energy density associated with an electric field is

$$u = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E}.$$

This is valid for linear dielectric materials. The total potential energy is

$$U = \int_\tau \mathbf{D}\cdot\mathbf{E}\,\mathrm{d}\tau.$$

When a capacitor is connected in series with a resistor and a source of constant emf, the voltage across the capacitor increases exponentially, with time constant RC, and the current decreases exponentially:

$$V_C = V_\mathrm{s}\left(1 - \exp\left(-\frac{t}{RC}\right)\right); \quad I = \frac{V_\mathrm{s}}{R}\exp\left(-\frac{t}{RC}\right).$$

Capacitors are widely used for energy storage.

Section 8.2 When an inductor is connected in series with a resistor and a source of emf, the voltage across the inductance and the current in the circuit change exponentially, with time constant L/R:

$$V_L = V_\mathrm{s}\exp\left(-\frac{Rt}{L}\right); \quad I = \frac{V_\mathrm{s}}{R}\left[1 - \exp\left(-\frac{Rt}{L}\right)\right].$$

The potential energy stored in an isolated inductance is $U = \tfrac{1}{2}LI^2$.

Alternatively, the magnetic potential energy can be associated with the magnetic field. The energy density associated with a magnetic field is

$$u = \tfrac{1}{2}\mathbf{B}\cdot\mathbf{H}.$$

The total magnetic potential energy is

$$U = \int_\tau \mathbf{B}\cdot\mathbf{H}\,\mathrm{d}\tau.$$

This is valid for linear magnetic materials. Magnetic fields produced by large currents in coils can be used for energy storage.

Section 8.3 In circuits that contain both an inductor and a capacitor, there can be an interchange of energy between the two components. Such circuits are used for pulse shaping and as oscillators, where the natural angular frequency is $\omega_n = 1/\sqrt{LC}$.

Book 2 Chapter 9

Section 9.1 Superconductivity was discovered in 1911, and in the century since then there have been many developments in knowledge of the properties of superconductors and the materials that become superconducting, in the theoretical understanding of superconductivity, and in the applications of superconductors.

Section 9.2 A superconductor has zero resistance to flow of electric current, and can sustain a current indefinitely. The magnetic flux remains constant in a completely superconducting circuit, since changes in the flux from the field applied to the circuit are balanced by changes to (persistent) currents induced in the circuit. For each superconductor there is a critical temperature T_c below which the material is superconducting.

Superconductors also exhibit perfect diamagnetism, with $\mathbf{B} = \mathbf{0}$ in the bulk of the material. The exclusion of magnetic field is known as the Meissner effect. An external magnetic field penetrates for a short distance into the surface of a superconducting material, and a current flows in the surface layer to screen the interior of the material from the applied field. Superconductivity is destroyed when the magnetic field strength exceeds a critical value for the material. The critical field strength falls to zero as the temperature is raised to the critical temperature. A superconducting specimen will have a critical current I_c above which the material reverts to the normal state. This critical current corresponds to the field strength exceeding the critical field strength in some region of the specimen.

Section 9.3 The two-fluid model of a superconductor regards some of the conduction electrons as behaving like normal electrons and some like superconducting electrons. For $T \ll T_\mathrm{c}$, all of the conduction electrons are superconducting electrons, but the proportion of superconducting electrons drops to zero at the critical temperature.

For a perfect conductor (which has $R = 0$), Maxwell's equations predict that the magnetic field cannot change, except in a thin surface layer. This does not predict the Meissner effect. The London equations are relationships between current density and magnetic field that are consistent with the Meissner effect:

$$\operatorname{curl} \boldsymbol{J}_\mathrm{s} = -\frac{n_\mathrm{s} e^2}{m}\mathbf{B} \quad \text{and} \quad \frac{\partial \mathbf{J}_\mathrm{s}}{\partial t} = \frac{n_\mathrm{s} e^2}{m}\mathbf{E}.$$

When combined with Maxwell's equations, they lead to the prediction that the magnetic field strength and the surface current decrease exponentially below the surface of a superconductor, over a characteristic distance called the penetration depth λ, which is typically tens of nanometres. The London equations are local relationships and therefore are strictly valid only when $\lambda \gg \xi$, where the coherence length ξ is the characteristic distance over which n_s varies.

Section 9.4 There are two types of superconductors, type-I and type-II. For a type-I material in the form of a thin specimen parallel to the field, there is an abrupt transition to the normal state at the critical field strength B_c. When the field is inclined to the surface of a type-I material, the material exists in the intermediate state over a range of field strengths below B_c. In this state there are thin layers of normal and superconducting material, with the proportion of normal material rising to unity at field strength B_c. In type-I materials, the coherence length ξ is greater than the penetration depth λ, and the surface energy of the boundary between superconducting and normal material is positive, which favours a course subdivision into regions of normal and superconducting material.

A type-II superconductor has two critical field strengths, $B_{\mathrm{c}1}$ and $B_{\mathrm{c}2}$, between which the material is in the mixed state. In this state the superconductor is threaded by thin cores of normal material, through which the magnetic field passes. The coherence length ξ is shorter than the penetration depth λ, and the surface energy of the boundary between superconducting and normal material is negative, which favours a fine subdivision into regions of normal and superconducting material. To take advantage of the high values of $B_{\mathrm{c}2}$ to produce high magnetic fields with superconducting magnets, it is essential to pin the normal cores to inhibit their motion.

Book 2 Chapter 10

Section 10.1 The theory of special relativity concerns the relationships between observations made by observers who use inertial frames of reference (i.e. frames in which Newton's first law holds true) that are in uniform relative motion. For inertial frames $\mathcal{F}$ and $\mathcal{F}'$ in standard configuration, the coordinates of an event observed at (t, x, y, z) in $\mathcal{F}$ are related to the coordinates (t', x', y', z') of the same event in $\mathcal{F}'$ by the Lorentz transformation (Equations 10.1 to 10.4. The effect of such a transformation is reversed by the inverse Lorentz transformation (Equations 10.6 to 10.9. Special relativity may be used to deduce the relativity of

simultaneity, the relativity of length (length contraction), and the relativity of intervals of time (time dilation). It may also be used to deduce transformation rules for the components of a velocity $\mathbf{U}$ (Equations 10.12 to 10.14 and for the components of a force $\mathbf{F}$ (Equations 10.17 to 10.19 , where $\mathbf{F}$ is defined as the rate of change of momentum, and momentum is defined by the relation $\mathbf{p} = m\mathbf{U}/\sqrt{1 - U^2/c^2}$.

Section 10.2 The charge q of a particle is an invariant quantity that takes the same value in all inertial frames. Other invariants are the (rest) mass m of a particle, the speed of light in a vacuum c, and the constants ε_0 and μ_0. Under a transformation between inertial frames $\mathcal{F}$ and $\mathcal{F}'$, the charge density ρ and the current density $\mathbf{J}$, in the combination $(c\rho, J_x, J_y, J_z)$, behave in the same way as the coordinates (ct, x, y, z). Under the same transformation, the partial derivatives $(\partial/\partial(ct)$, $\partial/\partial x$, $\partial/\partial y$, $\partial/\partial z)$ obey the inverse Lorentz transformation, and the electric and magnetic field components become intermingled as described by Equations 10.26 to 10.31.

Section 10.3 The principle of relativity demands that laws of physics should be form invariant under a transformation from one inertial frame to another. Unlike the laws of Newtonian mechanics, the laws of electromagnetism (including Maxwell's equations, the equation of continuity and the Lorentz force law) satisfy this requirement. The form invariance of electromagnetic laws can be demonstrated using the transformation rules for $\mathbf{F}$, $\mathbf{U}$, ρ, $\mathbf{J}$, $\mathbf{E}$, $\mathbf{B}$ and the relevant partial derivatives. This involves a great deal of algebra. The unified nature of the electromagnetic field can be made explicit by expressing the electromagnetic field as a tensor that incorporates the three electric and three magnetic field components.

Book 3

Book 3 Chapter 1

Section 1.1 The wave equation in one dimension is

$$\frac{\partial^2}{\partial z^2}A(z,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}A(z,t) = 0,$$

where v is the speed of the waves. The plane wave solutions of this equation can be expressed in terms of linear combinations of sinusoidal functions, representing waves travelling in either the positive or negative z-direction. The general solutions proposed by d'Alembert have the form $g(z - vt)$ and $g(z + vt)$, and represent waves with fixed profiles travelling in the positive and negative z-directions, respectively.

Section 1.2 A three-dimensional wave equation,

$$\nabla^2\mathbf{B}(\mathbf{r},t) = \varepsilon_0\mu_0\frac{\partial^2\mathbf{B}(\mathbf{r},t)}{\partial t^2},$$

can be extracted from Maxwell's equations. In a vacuum, a universal speed of propagation emerges from the algebra: $c = 1/\sqrt{\varepsilon_0\mu_0}$.

Section 1.3 Far from a source, the solutions of the wave equation are plane waves. When only a single frequency is present, we talk about monochromatic waves. Gauss's law and the no-monopole law impose a requirement for the electric and magnetic fields to be transverse to the propagation direction. Faraday's law and the Ampère–Maxwell law ensure that the electric and magnetic fields are perpendicular to each other and that the ratio of their amplitudes is $c = 1/\sqrt{\varepsilon_0\mu_0}$. The wave solutions of Maxwell's equations consist of oscillating electric and magnetic fields synchronized exactly in phase. The result of the coupling of the electric and magnetic fields is that electromagnetic radiation can propagate through empty space.

Complex exponential notation provides a compact way of representing linearly polarized, monochromatic plane waves, and is particularly suitable for working with the differentials in Maxwell's equations. The fields are the real parts of complex expressions like

$$\mathbf{E}(z,t) = E_{0x}\exp[\mathrm{i}(kz - \omega t + \phi)]\,\mathbf{e}_x.$$

The complex notation is easily generalized to describe monochromatic waves travelling in arbitrary directions and with arbitrary polarizations. Linear combinations of electromagnetic waves, polarized at right angles to each other, with equal amplitudes, constitute a linearly polarized wave if they are in phase and a circularly polarized wave if they differ in phase by $\pi/2$.

Section 1.4 The energy flux density in an electromagnetic wave is given by the Poynting vector, $\mathbf{N} = \mathbf{E}_{\text{phys}} \times \mathbf{H}_{\text{phys}}$. The time-average of the Poynting vector is

$$\overline{\mathbf{N}} = \tfrac{1}{2}\varepsilon_0(E_{0x})^2c\,\mathbf{e}_z.$$

Book 3 Chapter 2

Section 2.1 Electric and magnetic fields that are associated with radiation from a point source must fall off in strength as r^{-1} in the absence of absorption, in order that energy is conserved as radiation spreads outwards.

Section 2.2 We can construct a magnetic field solution of Maxwell's equations for an oscillating current element (a Hertzian dipole) using the Biot–Savart field law to guide the dependence on ϕ, θ, current amplitude I_0 and dipole length δl; adding a term with r^{-1} scaling to the inherent r^{-2}-dependence of the Biot–Savart law to conform with the requirements of radiation fields; we must also retard the time $(t \to t - r/c)$ to allow for propagation of information from the source to the observer.

Section 2.3 Maxwell's equations are used to calibrate and validate the predicted solution for the field from a Hertzian dipole. In this process, the **B** and **E** fields are confirmed as having terms with various radial-dependences.

Section 2.4 The terms with r^{-1}-dependences predominate when $r \gg \lambda$, and these terms form the radiation fields:

$$\mathbf{B}^{(\mathrm{rad})} = \frac{\omega\eta}{c}\frac{\sin\theta}{r}\sin(kr - \omega t)\,\mathbf{e}_\phi,$$

$$\mathbf{E}^{(\mathrm{rad})} = \omega\eta\frac{\sin\theta}{r}\sin(kr - \omega t)\,\mathbf{e}_\theta,$$

where $\eta = \mu_0 I_0\,\delta l/4\pi$.

The radiation fields satisfy a spherical wave equation. The fields correspond with electromagnetic waves that have a $\sin\theta$ angular-dependence, so that no power is radiated along the axis of the dipole and maximum power is radiated in the equatorial plane of the dipole. The total power radiated from an oscillating *current* dipole is

$$\overline{W} = \frac{\mu_0\omega^2 I_0^2(\delta l)^2}{12\pi c},$$

whereas in terms of an oscillating *charge*, the total radiated power is

$$\overline{W} = \frac{\mu_0 p_0^2\omega^4}{12\pi c} = \frac{p_0^2\omega^4}{12\pi\varepsilon_0 c^3}.$$

Section 2.5 The scattering of sunlight by the Earth's atmosphere can be understood in terms of radiation from dipoles induced in atoms and molecules by the electric field of electromagnetic radiation from the Sun. Owing to the ω^4 frequency-dependence of radiation scattered by an induced dipole, blue light is scattered much more effectively than red. This accounts for the blue appearance of clear skies. The $\sin\theta$-dependence of the amplitude of electromagnetic waves radiated from induced dipoles causes partial polarization of sky light, and this is strongest when viewed perpendicular to the incident sunlight.

Book 3 Chapter 3

Section 3.1 Expressions for electromagnetic wave propagation in LIH dielectrics (with $\mu = 1$) can be obtained from those for free space by replacing ε_0 with $\varepsilon\varepsilon_0$ and replacing c with c/n, where the refractive index, $n = \sqrt{\varepsilon}$, is the ratio of the speed of propagation in free space to that in the dielectric. In particular, the wavenumber and angular frequency are linked by

$$k = n\omega/c,$$

and the time-averaged energy flux per unit area is

$$\overline{\mathbf{N}} = \tfrac{1}{2}\varepsilon\varepsilon_0 E_0^2\frac{c}{n}\,\widehat{\mathbf{k}}.$$

Section 3.2 In the absence of free charges and currents in the bulk or at surfaces, the boundary conditions on the time-dependent fields **E**, **D**, **B** and **H** at the interface between two different dielectric media are the same as for static fields, namely that the parallel components of **E** and **H**, and the perpendicular components of **D** and **B**, are continuous at the boundary.

Section 3.3 Transmission and reflection of electromagnetic waves at an interface between two different dielectric media can be quantified in terms of the amplitude transmission and reflection ratios, which are defined as the ratios of the amplitudes of the transmitted and reflected fields to the amplitude of the incident

field, or by the transmittance and reflectance, which are defined as the ratios of the transmitted and reflected powers to the incident power. Expressions for these quantities can be determined by applying the boundary conditions at the interface.

Section 3.4 The laws of reflection and Snell's law can be derived from a consideration of travelling wave solutions of Maxwell's equations at a dielectric boundary.

The behaviour of an electromagnetic wave incident at any angle onto an interface between two different dielectric media can be built up from a linear combination of two special cases: one in which the wave is polarized in the scattering plane, and one in which it is polarized normal to the scattering plane. Applying the standard boundary conditions to these two cases leads to the Fresnel equations for the amplitude transmission and reflection ratios. For light incident at the Brewster angle, $\theta_\mathrm{B} = \tan^{-1}(n_2/n_1)$, the component of the wave that is polarized in the scattering plane is completely transmitted, so the reflected wave is wholly polarized normal to the scattering plane.

If light passes from one dielectric to another of lower refractive index (e.g. from glass to air), then for incidence at the so-called critical angle, $\theta_\mathrm{crit} = \sin^{-1}(n_2/n_1)$, the transmitted wave emerges parallel to the interface, i.e. $\theta_\mathrm{t} = \pi/2$. When the angle of incidence is greater than the critical angle, there is no real solution for the angle of refraction, and the wave is said to undergo total internal reflection.

The propagation of electromagnetic energy in a slab, rod or fibre of dielectric, immersed in a second dielectric with lower refractive index, can result in energy being guided along the high index material and along the boundaries. This is the basis of optical fibre telecommunications.

Section 3.5 Results derived for LIH media are appropriate for the study of optics in simple transparent dielectrics such as glass and water. Materials that are non-linear, anisotropic or inhomogeneous have different properties.

Book 3 Chapter 4

Section 4.1 Dispersion and absorption degrade light pulses as they travel along optical fibres. This establishes the need for a model that accounts for these processes.

Section 4.2 The classical model of a dielectric is based on a dynamical description of electrons bound to atoms. The model leads to a dielectric function that expresses the relative permittivity as a complex, frequency-dependent quantity, $\varepsilon_\mathrm{real}(\omega) + \mathrm{i}\,\varepsilon_\mathrm{imag}(\omega)$. The real part of the dielectric function is almost constant except in the neighbourhood of characteristic resonant frequencies, where it changes rapidly. In the same frequency band, the imaginary part of the dielectric function rises from virtually zero to a peak. The refractive index is similarly a complex function of frequency.

Section 4.3 The frequency-dependence of the real part of the dielectric function leads to dispersion of waves of different frequencies owing to their different speeds of propagation. Angular dispersion arises when electromagnetic radiation is refracted as it crosses an interface between two dielectric materials. Dispersion also spreads radiation energy as it propagates through the bulk of a dielectric. Linear combinations of plane waves with different frequencies do not propagate as a fixed shape in a dispersive medium; that is, the wave equation does *not* have solutions of the d'Alembert form except in the special case where the contributing plane waves all travel with the same speed. Information is transmitted by modulated waves and wave packets rather than continuous sinusoidal waves. Energy and information propagate at the group speed $\mathrm{d}\omega/\mathrm{d}k$, whereas the component waves travel at the phase speed ω/k.

Section 4.4 The imaginary part of the dielectric function is associated with the loss of energy from an electromagnetic wave as it traverses a dielectric medium. The amplitude of the fields decreases exponentially with distance as $\exp(-k_\mathrm{imag}z)$, and the energy as $\exp(-2k_\mathrm{imag}z)$.

Section 4.5 Measurements of refractive index yield dielectric data for water and for silica. In both cases there are features that can be recognized from the simple model of Section 4.2. The overall picture is complicated by the presence of a variety of absorption pathways. The attenuation of the signal in a dielectric is also in part due to scattering from impurities and inhomogeneities.

Book 3 Chapter 5

Section 5.1 The electric field of an electromagnetic wave in a conductor will drive current. A simple model of a conductor accounts for the motion of unbound electrons in a similar way to the model for bound electrons in a dielectric in Chapter 4. This leads to an effective relative permittivity function $\varepsilon_\mathrm{eff}(\omega)$ that exhibits three characteristic styles of behaviour. Well above the frequency $\omega_\mathrm{p} = \sqrt{n_\mathrm{e}e^2/m\varepsilon_0}$, the unbound

electrons in a conductor cannot respond and electromagnetic waves pass unhindered. In the range between ω_p and the frequency of collisions between electrons and the lattice ($1/\tau_\mathrm{c}$), electromagnetic waves will not propagate in a conductor but evanesce. At frequencies well below $1/\tau_\mathrm{c}$, that is, where $\omega\varepsilon_0/\sigma \ll 1$, conduction currents dominate and energy is strongly absorbed from an electromagnetic field. Electric and magnetic fields effectively diffuse into a conductor at these frequencies:

$$\nabla^2\mathbf{E} = \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}.$$

The depth to which electromagnetic fields penetrate a conductor is characterized by the skin depth:

$$\delta = \frac{1}{k_\mathrm{imag}} = \sqrt{\frac{2}{\mu_0\sigma\omega}}.$$

Section 5.2 Transmission and reflection of waves at the boundary of a conductor is determined by the boundary conditions for the fields. For a wave incident from material 1 onto the surface of conducting material 2, the conditions are:

$$B_{1\perp} = B_{2\perp}; \quad D_{2\perp} - D_{1\perp} = \sigma_\mathrm{f}; \quad E_{1\|} = E_{2\|}; \quad H_{2\|} - H_{1\|} = i_\mathrm{s},$$

where σ_f is the free surface charge density and i_s is the free surface current per unit length.

Section 5.3 When electromagnetic waves are normally incident on the surface of a conductor, the small fraction of energy transmitted through the surface is absorbed within the skin depth due to Joule heating. In conductors where the skin depth is small compared with other dimensions, it is reasonable to invoke a perfect conductor model, in which $\mathbf{E}$ and $\mathbf{B}$ are zero within the material, and the induced current and charge are restricted exactly to the surface.

The boundary conditions at the surface of a perfect conductor are $E_\| = 0$ and $B_\perp = 0$. Perfect conductors reflect all incident electromagnetic waves. Electric fields undergo a phase change of π on reflection from a perfect conductor, but the phase of magnetic fields is not affected by reflection.

Section 5.4 When a plane wave is incident on a perfect conductor, the superposition of the incident and reflected waves produces a standing wave pattern perpendicular to the surface and a travelling wave parallel to the surface. For the region between two parallel perfectly-conducting planes, the zero-electric-field boundary condition allows only wave modes that correspond to angles of incidence θ such that

$$\cos\theta = \left(\frac{m\pi}{k_0 a}\right), \quad \text{where } m = 1, 2, 3, \ldots.$$

Each of these modes has a standing wave pattern perpendicular to the planes, with wavenumber $k_\mathrm{c} = k_0\cos\theta$, or wavelength $\lambda_\mathrm{c} = 2a/m$, and a travelling wave parallel to the planes, with wavenumber $k_\mathrm{gw} = k_0\sin\theta$. Waves with free space wavelength λ_0 greater than the cut-off wavelength $\lambda_\mathrm{c} = 2a/m$, or equivalently waves with frequency less than the cut-off frequency $f_\mathrm{c} = (m/2a)c$, are evanescent. For transverse electric (TE) modes, the electric field is perpendicular to the direction of propagation, but the magnetic field has a component in the direction of propagation. Energy is transmitted parallel to the planes at the group speed,

$$v_\mathrm{group} = \frac{\mathrm{d}\omega}{\mathrm{d}k_\mathrm{gw}} = \frac{c^2 k_\mathrm{gw}}{\omega}.$$

The phase speed, $v_\mathrm{phase} = \omega/k_\mathrm{gw} = \omega/k_0\sin\theta$, exceeds c, but the group speed is always less than c.

Section 5.5 Waveguides are hollow conductors within which there are solutions of Maxwell's equations that satisfy the boundary conditions and correspond to waves propagating along the guide. TE modes in a rectangular waveguide are characterized by two integers, m and n, which indicate the number of half cycles of standing wave that fit into the long and short dimensions of the guide cross-section. For each mode there is a cut-off frequency below which the mode cannot propagate, and for the TE_{mn} mode this is given by

$$f_{mn} = c\sqrt{\left(\frac{m}{2a}\right)^2 + \left(\frac{n}{2b}\right)^2}.$$

Book 3 Chapter 6

Section 6.1 Plasmas are ionized gases comprising electrons, ions and neutral atoms and molecules. They are neutral overall, and the density of free charges is such that there is strong coupling between the charges; as a result, the properties of plasmas are markedly different from those of neutral gases. Thermal energy, radiation and electrical energy can all contribute to sustaining the ionized condition of a plasma. Plasmas

exist in stars, in the interstellar medium and around the Earth in the ionosphere and the magnetosphere, as well as in artificial and natural electrical discharges. The effective temperature of the particles in a plasma, especially the electrons, can be equivalent to tens of thousands of kelvins, though heavier particles are often considerably cooler.

The characteristic resonance frequency for oscillations in a plasma, known as the plasma frequency, is given by

$$\omega_{\mathrm{p}} = \sqrt{\frac{n_{\mathrm{e}} e^2}{m\varepsilon_0}}.$$

Treating a plasma as a linear isotropic homogeneous gaseous conductor, and neglecting thermal motion of the constituent particles, leads to an expression for the relative permittivity function that is analogous to that of an ordinary conductor. If the background gas pressure is low enough, that is, $1/\tau_{\mathrm{c}} \ll \omega$, then

$$\varepsilon_{\mathrm{eff}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega^2}.$$

Electromagnetic waves in LIH plasmas can propagate only if $\omega > \omega_{\mathrm{p}}$. At lower frequencies, the solutions of Maxwell's equations are evanescent.

Section 6.2 The model developed in Section 6.1 is applicable to the ionospheric plasma. The ionosphere reflects normally-incident electromagnetic waves at a height where the plasma frequency equals the wave frequency. The plasma frequency corresponding to the maximum electron number density in the ionosphere is about 10 MHz, so waves above this frequency can be transmitted through the ionosphere into space. The ionosphere is a near perfect reflector of obliquely-incident electromagnetic waves for frequencies up to about 30 MHz, and this accounts for the long range of radio transmissions between 5 and 30 MHz.

Section 6.3 A steady magnetic field makes a plasma anisotropic. The magnetic component of the Lorentz force leads to a sensitivity to circularly polarized waves, with left- and right-handed polarizations provoking a different response from the plasma. The relative permittivity functions for circularly polarized waves travelling parallel to the magnetic field are

$$\varepsilon^{\mathrm{RH}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega - \omega_{\mathrm{c}})} \quad \text{and} \quad \varepsilon^{\mathrm{LH}}(\omega) = 1 - \frac{\omega_{\mathrm{p}}^2}{\omega(\omega + \omega_{\mathrm{c}})},$$

in which ω_{c} is the electron cyclotron frequency, eB/m.

A plasma in a steady magnetic field is able to support propagating electromagnetic waves at frequencies below the plasma frequency (the lower limit for propagation in LIH plasmas). Whistler waves, which propagate in the Earth's magnetosphere, are an example of the RH polarized mode at low frequency.

Section 6.4 Various methods exist for producing laboratory plasmas that are sustained by energy absorbed from the low-frequency mode of RH circularly polarized electromagnetic waves. Electron cyclotron resonance can be used directly to couple microwave energy into a plasma in a magnetic field. Another method couples radio frequency electromagnetic waves into the helicon mode of a magnetized plasma. As the wave travels, its energy is transferred to the plasma's particles through collisional damping.

Book 3 Chapter 7

Section 7.1 The mammalian cornea is made up of a composite dielectric formed from collagen fibrils in a transparent matrix medium. The fibrils are packed into lamellae, and within each lamella the fibrils are essentially parallel to each other and to the lamella surface. The spacing between fibril centres is about one tenth of a typical mid-range optical wavelength. The thickness of a lamella is about five wavelengths, and the thickness of the cornea is about twelve hundred wavelengths.

Owing to differences in relative permittivity between fibrils and the matrix, an isolated fibril scatters incident electromagnetic radiation. The scattering cross-section is, however, very small; a fibril scatters less than 0.1% of the radiation that is incident upon it. Each layer of fibrils in a lamella effectively forms a diffraction grating, but since the spacing between fibrils is less than one wavelength, only zero-order diffraction occurs. This means that overall the fibrils are not 'visible' — they do not, as a whole, scatter light, because they are arranged as arrays of almost identical, low cross-section scatterers, packed at sub-wavelength separation. As a consequence, the cornea is highly transparent.

Section 7.2 A pair of parallel wires can guide TEM waves through surrounding dielectric material. A straight-sided hairpin can therefore be modelled as a resonant structure on which TEM standing wave modes can be excited. The fundamental resonance occurs when the length of the hairpin corresponds to one quarter of the wavelength of a TEM wave.

When surrounded by an ordinary dielectric material, the resonant frequency of the hairpin is decreased. However, in a plasma, the presence of free electrons gives rise to a relative permittivity ($\varepsilon = 1 - f_{\mathrm{p}}^2/f^2$) that causes the resonance to shift to higher frequency. The frequency shift can be simply related to the number density of free electrons in the plasma.